

# Distribution of Maximum Loss for Fractional Brownian Motion

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**Abstract** In finance, the price of a volatile asset can be modeled using fractional Brownian motion (fBm) with Hurst parameter  $H > \frac{1}{2}$ . The Black-Scholes model for the values of returns of an asset using fBm is given as,

$$Y_t = Y_0 \exp((r + \mu)t + \sigma B_t^H), \quad t \geq 0$$

where  $Y_0$  is the initial value,  $r$  is constant interest rate,  $\mu$  is constant drift and  $\sigma$  is constant diffusion coefficient of fBm, which is denoted by  $(B_t^H)$  where  $t \geq 0$ . Black-Scholes model can be constructed with some Markov processes such as Brownian motion. The advantage of modeling with fBm to Markov processes is its capability of exposing the dependence between returns. The real life data for a volatile asset display long-range dependence property. For this reason, using fBm is a more realistic model compared to Markov processes. Investors would be interested in any kind of information on the risk in order to manage it or hedge it. The maximum possible loss is one way to measure highest possible risk. Therefore, it is an important variable for investors. In our study, we give some theoretical bounds on the distribution of maximum possible loss of fBm. We provide both asymptotical and strong estimates for the tail probability of maximum loss of standard fBm and fBm with drift and

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diffusion coefficients. In the investment point of view, these results explain, how large values of possible loss behave and its bounds.

**Keywords** First keyword · Second keyword · More

## 1 Introduction

In finance, the price of one share of the risky asset, is modeled using fractional Brownian motion (fBm) with Hurst parameter  $H \in [1/2, 1)$  in order to display long-range dependence. Hence, our results are on fractional Brownian motion with Hurst parameter  $H \in [1/2, 1)$ .

There are several stochastic integral representations that have been developed for fBm. For example,

$$\begin{aligned} B_t^H &= \frac{1}{\Gamma(H + 1/2)} \int_{\mathbb{R}} ((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}) dW_s \\ &= \frac{1}{\Gamma(H + 1/2)} \left( \int_{-\infty}^0 ((t-s)^{H-1/2} - (-s)^{H-1/2}) dW_s \right. \\ &\quad \left. + \int_0^t (t-s)^{H-1/2} dW_s \right) \end{aligned} \quad (1)$$

is a fBm with Hurst parameter  $H \in (0, 1)$ , where  $W$  is a Wiener process.

Let  $H$  be a constant in the interval  $(0, 1)$ . A (standard) fBm  $\{B_t^H : t \geq 0\}$  with Hurst parameter  $H$  is a continuous and centered Gaussian process with covariance function

$$E[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \quad (2)$$

and  $B_0^H = 0$ . It follows that  $B^H$  has stationary increments, that is  $B_{t+s}^H - B_s^H$  has the same law as  $B_t^H$ , for  $s, t \geq 0$ .

For  $H = 1/2$ , the process  $(B_t^H)_{t \geq 0}$  corresponds to a standard Brownian motion, in which the increments are independent. By definition, the covariance between the increments  $B^H(t+h) - B^H(t)$  and  $B^H(s+h) - B^H(t)$  with  $s+h \leq t$  and  $t-s = nh$  is

$$\rho_H(n) = \frac{1}{2} h^{2H} [(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}].$$

We observe that two increments of the form  $B^H(t+h) - B^H(t)$  and  $B^H(t+2h) - B^H(t+h)$  are positively correlated for  $H > 1/2$ , and they are negatively correlated for  $H < 1/2$ .

Since the covariance function of fBm is homogeneous of order  $2H$ , [2] fBm possesses the self-similarity property, that is, for any constant  $c > 0$ ,

$$(B_{ct}^H)_{t \geq 0} \stackrel{law}{=} (c^H B_t^H)_{t \geq 0}.$$

The aim of the present paper is to find the distribution of maximum possible loss of fBm, which is one way of measuring investors risk. In order to obtain

results related to this distribution, we are also interested in the supremum, the infimum, and the range variables of fBm.

Let  $(\Omega, \mathfrak{F}, P)$  be a probability space and let  $B^H$  be fractional Brownian motion with Hurst parameter  $H \in [1/2, 1)$ . We introduce the following notation.

- Let  $I_t^H := \inf_{0 \leq v \leq t} B_v$  denote the *infimum* of fractional Brownian motion up to time  $t$ .
- Let  $S_t^H := \sup_{0 \leq v \leq t} B_v$  denote the *supremum* of fractional Brownian motion up to time  $t$ .
- Let  $R_t^H = S_t^H - I_t^H$ , called the *range* of fractional Brownian motion up to time  $t$ .
- The *maximum loss* of fractional Brownian motion before time  $t$  is defined as

$$M_t^{H,-} := \sup_{0 \leq u \leq v \leq t} (B_u - B_v) = \sup_{0 \leq v \leq t} \left( \sup_{0 \leq u \leq v} (B_u - B_v) \right)$$

More generally, the above definitions can be repeated for a non-standard fBm. Let  $\mu$  be drift parameter taking real values other than 0 and similarly  $\sigma > 0$  be a real valued diffusion coefficient other than 1 for fBm with drift defined as  $Y_t := \mu t + \sigma B_t$ .

## 2 Bounds on the distribution of maximum loss

In this section we introduce some new bounds on the expected value of maximum loss of fBm and on the distribution of maximum loss of fBm. These results are already useful for investors as they are. Also, in later sections they will be useful for finding the asymptotic distribution of maximum loss.

**Theorem 1** *For fBm up to time  $a$  with Hurst parameter  $H > \frac{1}{2}$ , and for  $y > 0$ , we have*

$$\frac{\sqrt{2}a^H}{2\sqrt{\pi}} \leq E(M_a^{H,-}) \leq \frac{2\sqrt{2}a^H}{\sqrt{\pi}}$$

and

$$P(M_a^{H,-} > y) < P(R_a^H \geq y) \leq \frac{2\sqrt{2}a^H}{y\sqrt{\pi}}$$

**Proof:**  $E(S_a^H) \leq \frac{\sqrt{2}}{\sqrt{\pi}} * a^H$  is known [3] and by the Markov's inequality an upper bound for the distribution of the supremum is found, that is for  $x > 0$ ,  $P(S_a^H > x) \leq \frac{\sqrt{2}a^H}{x\sqrt{\pi}}$ . Then, by the symmetry property of centered Gaussian processes one can show that  $E(I_a^H) \geq -\frac{\sqrt{2}}{\sqrt{\pi}} * a^H$

Combining the results given above we find an upper bound for the expected value of range,  $R_t^H$ , that is  $E(R_a^H) \leq \frac{2\sqrt{2}}{\sqrt{\pi}} * a^H$ . Furthermore, by Markov's inequality we see that for  $y > 0$ ,  $P(R_a^H \geq y) \leq \frac{2\sqrt{2}a^H}{y\sqrt{\pi}}$ .

Clearly one can see that,

$$I_t^H := - \inf_{0 \leq v \leq t} X_v \leq \sup_{u \leq v \leq t} \sup_{0 \leq u \leq v} (X_u^H - X_v^H) = M_t^{H,-} \leq R_t^H$$

holds for all  $t$ .

Hence

$$-E(I_a^H) \leq E(M_a^{H,-}) \leq E(R_a^H) \leq \frac{2\sqrt{2}a^H}{\sqrt{\pi}} \quad (3)$$

is obtained, and by Markov's inequality we get

$$P(M_a^{H,-} > y) < P(R_a^H \geq y) \leq \frac{2\sqrt{2}a^H}{y\sqrt{\pi}}$$

We have also noticed that,  $\frac{\sqrt{2}a^H}{2\sqrt{\pi}} \leq E(S_a^H) \leq \frac{\sqrt{2}a^H}{\sqrt{\pi}}$  [8]. These bounds are obtained using  $H = 1$  and  $H = 1/2$  in Sudakov-Fernique inequality [1, Theorem II.2.9].

Furthermore, because  $E(S_a^H)$  equals  $-E(I_a^H)$  using Equation (3) we obtain,

$$\frac{\sqrt{2}a^H}{2\sqrt{\pi}} \leq E(M_a^{H,-}) \leq \frac{2\sqrt{2}a^H}{\sqrt{\pi}}.$$

□

### 3 Asymptotical distribution of maximum loss of fractional Brownian motion

The theorem in this section gives the asymptotic result for the probability distribution of maximum loss of fBm. We first start with scaling property of loss process which we denote as  $X_v := \sup_{0 \leq u \leq v} (B_u - B_v)$ ,  $v > 0$ . Our approach is similar to the large deviations technique used for queueing systems modeled by fractional Brownian motion, [5] and [8].

We define the loss process  $X$  by  $X_v = \sup_{0 \leq u \leq v} (B_u - B_v)$ .

**Proposition 1** *The loss process  $X$  is self-similar and each  $X_v$  has the same distribution as the supremum  $S_v$  for every  $v \geq 0$ .*

**Proof:** The self-similarity of fBm corresponds to  $\{B_{au} : u \geq 0\} \stackrel{d}{=} \{a^H B_u : u \geq 0\}$  for every  $a > 0$ . It follows that

$$\{B_{au} - B_{av} : 0 \leq u \leq v, v \geq 0\} \stackrel{d}{=} \{a^H (B_u - B_v) : 0 \leq u \leq v, v \geq 0\}.$$

Therefore, we get

$$\{X_{av} : v \geq 0\} \stackrel{d}{=} \{a^H X_v : v \geq 0\}$$

by the definition  $X_v = \sup_{0 \leq u \leq v} (B_u - B_v)$ .

On the other hand, since fractional Brownian motion has stationary increments, the collections  $\{B_u - B_v : 0 \leq u \leq v\}$  and  $\{-B_{v-u} : 0 \leq u \leq v\}$  have the same probability law for fixed  $v$ . Both are 0 mean Gaussian processes with covariance function

$$r(u, u') = 1/2 [ |v - u|^{2H} + |v - u'|^{2H} - |u - u'|^{2H} ].$$

Since the supremum of the two collections will also have the same distribution and  $\{-B_{v-u} : 0 \leq u \leq v\} \stackrel{d}{=} \{B_u : 0 \leq u \leq v\}$ , we get  $X_v \stackrel{d}{=} S_v^H$ .  $\square$

Now, let us use  $\bar{\Phi}$  to denote cumulative distribution function of Normal distribution and  $\bar{\Phi}$  for its complement distribution

**Proposition 2** *For all  $x \in R_+$ , we have*

$$P(M_t^{H,-} > x) \geq \bar{\Phi}(x/t^H).$$

**Proof:** As a result of Proposition 1, for  $0 \leq v \leq t$ , we see that,

$$P(X_v > x) = P(\sup_{0 \leq u \leq v} B_u > x) \geq \sup_{0 \leq u \leq v} P(B_u > x) = \sup_{0 \leq u \leq v} \bar{\Phi}(x/u^H) = \bar{\Phi}(x/v^H). \quad (4)$$

Similarly, we observe

$$P(M_t^{H,-} > x) = P(\sup_{0 \leq v \leq t} X_v > x) \geq \sup_{0 \leq v \leq t} P(X_v > x) = \sup_{0 \leq v \leq t} \bar{\Phi}(x/v^H) = \bar{\Phi}(x/t^H) \quad (5)$$

**Theorem 2** *For the maximum loss  $M_t^{H,-}$  of fBm and  $x > 0$ , we have*

$$\lim_{x \rightarrow \infty} \frac{1}{x^2} \log P(M_t^{H,-} > x) = -\frac{1}{2t^{2H}}$$

**Proof:** We start with finding a lower bound for limit infimum of the logarithm of the distribution of maximum loss. Combining Proposition 2 with

$$\lim_{x \rightarrow \infty} (1/x^2) \log \bar{\Phi}(x/t^H) = -(2t^{2H})^{-1}$$

(Adler, 1990, sf.42), we obtain,

$$\liminf_{x \rightarrow \infty} \frac{1}{x^2} \log P(M_t^{H,-} > x) \geq \lim_{x \rightarrow \infty} \frac{1}{x^2} \log \bar{\Phi}(x/t^H) = -\frac{1}{2t^{2H}} \quad (6)$$

We continue with finding an upper bound for the limit supremum. Note that maximum loss  $M_t^{H,-}$  has two dimensional index set  $T := \{(u, v) : 0 \leq u \leq v \leq t\}$ , and it is a centered Gaussian process. Based on the fact that  $T$  is a separable metric space we use Borel's inequality given in [1, Theorem II.2.1] for finding the supremum of this process. By the continuity property of the paths of fractional Brownian motion,  $B_u - B_v$ , is bounded on  $T$ . And by Borel's inequality or directly from Theorem 1 we see that  $\eta := E[\sup_{0 \leq u \leq v \leq t} (B_u - B_v)]$  is finite.

As a result of Borel's inequality

$$P(M_t^{H,-} > x) = P\left(\sup_{0 \leq u \leq v \leq t} (B_u - B_v) > x\right) \leq 2e^{-\frac{1}{2}(x-\eta)^2/t^{2H}}, \quad x > \eta$$

can be written. Here,  $\sup_{0 \leq u \leq v \leq t} E(B_u - B_v)^2 = t^{2H}$  is used.

This shows,

$$\limsup_{x \rightarrow \infty} \frac{1}{x^2} \log P(M_t^{H,-} > x) \leq \lim_{x \rightarrow \infty} \left[ \frac{\log 2}{x^2} - \frac{1}{x^2} \frac{(x-\eta)^2}{2t^{2H}} \right] = -\frac{1}{2t^{2H}} \quad (7)$$

which completes the proof.  $\square$

In [6] it was shown that the result given in Theorem 2 for centered, Gaussian sequence of random variables which was mentioned in [1, pg.43]. However, in order for us to use this result, it must be shown for general metric space  $T$ , similar to the proof of Borel's inequality. In stead of showing this generalization, we preferred giving the proof for  $M_t^{H,-}$ . As a conclusion, we see that the increase in the logarithm of distribution of  $M_t^{H,-}$  behaves same as the asymptotic increase of distribution of  $B_t$ . Here, the important property of  $B_t$  is, it has the same distribution as the random variable with the highest variation among the random variables  $\{B_u - B_v : 0 \leq u \leq v \leq t\}$  which is  $B_0 - B_t = -B_t$ .

#### 4 Asymptotical Estimate for the Tail Probability

In the next theorem, we generalize the previous result for fractional Brownian motion  $Y$  with drift and diffusion coefficient. Recall  $Y_t = \mu t + \sigma B_t$  for  $\mu \in R$ ,  $\sigma > 0$ .

**Theorem 3** *The maximum loss process defined by  $M_t^{H,-} := \sup_{0 \leq u \leq v \leq t} (Y_u - Y_v)$  satisfies*

$$\lim_{x \rightarrow \infty} \frac{1}{x^2} \log P(M_t^{H,-} > x) = -\frac{1}{2\sigma^2 t^{2H}}$$

where  $Y$  is a fractional Brownian motion with drift and diffusion coefficients.

**Proof:** For all  $x \in R_+$ , the trivial lower bound is given by

$$P(M_t^{H,-} > x) = P\left(\sup_{0 \leq u \leq v \leq t} (Y_u - Y_v) > x\right) \geq \sup_{0 \leq u \leq v \leq t} P(Y_u - Y_v > x).$$

Since  $Y$  has stationary increments, we have

$$P(Y_u - Y_v > x) = P(-Y_{v-u} > x) = \bar{\Phi}((x + \mu(v-u))/(\sigma(v-u)^H)).$$

Therefore, the following hold

$$\sup_{0 \leq u \leq v \leq t} P(Y_u - Y_v > x) = \sup_{0 \leq u \leq v \leq t} \bar{\Phi} \left( \frac{x + \mu(v-u)}{\sigma(v-u)^H} \right) = \sup_{0 \leq v \leq t} \bar{\Phi} \left( \frac{x + \mu v}{\sigma v^H} \right)$$

Since  $\bar{\Phi}$  is a decreasing function, we find  $v \in [0, t]$  that minimizes  $f(v) = (x + \mu v)/(\sigma v^H)$ . The critical value of  $f$  is  $v^* = \frac{x^H}{\mu(1-H)}$ . We check if  $v^* \in [0, t]$  or not as below

- If  $\mu > 0$  and  $x < t\mu(1-H)/H$ , then the minimum value of  $f$  is obtained at  $v^* \in [0, t]$ .
- If  $\mu > 0$  and  $x > t\mu(1-H)/H$ , then the minimum of  $f$  is obtained at  $t$ .
- If  $\mu < 0$ , then the minimum of  $f$  occurs at  $t$ .

As a result, we have  $P(M_t^{H,-} > x) \geq \bar{\Phi}((x + \mu t)/(\sigma t^H))$  for large  $x > 0$ . We get

$$\liminf_{x \rightarrow \infty} \frac{1}{x^2} \log P(M_t^{H,-} > x) \geq \lim_{x \rightarrow \infty} \frac{1}{x^2} \log \bar{\Phi} \left( \frac{x + \mu t}{\sigma t^H} \right) = -\frac{1}{2\sigma^2 t^{2H}}.$$

On the other hand, we use Borel inequality which applies to a mean zero Gaussian process in order to find the limit supremum [1, Theorem II.2.1]. By definition,  $Y_u - Y_v = \sigma(B_u - B_v) + \mu(u - v)$  and

$$\begin{aligned} \sup_{0 \leq u \leq v \leq t} (Y_u - Y_v) > x &\Leftrightarrow \exists (u, v) \in t, \sigma(B_u - B_v) + \mu(u - v) > x \\ &\Leftrightarrow \exists (u, v) \in t, \frac{\sigma(B_u - B_v)}{x - \mu(u - v)} > 1 \\ &\Leftrightarrow \sup_{0 \leq u \leq v \leq t} \frac{\sigma(B_u - B_v)}{x - \mu(u - v)} > 1 \end{aligned}$$

where we assume  $x > \mu t$ . Making a change of variable  $(u, v) \rightarrow (u/x, v/x)$  and using the self-similarity of standard fractional Brownian motion, we get

$$\begin{aligned} P \left( \sup_{0 \leq u \leq v \leq t} (Y_u - Y_v) > x \right) &= P \left( \sup_{0 \leq u \leq v \leq t} \frac{\sigma(B_u - B_v)}{x - \mu(u - v)} > 1 \right) \\ &= P \left( \sup_{0 \leq u \leq v \leq \frac{t}{x}} \frac{\sigma(B_{xu} - B_{xv})}{x - \mu x(u - v)} > 1 \right) \\ &= P \left( \sup_{0 \leq u \leq v \leq \frac{t}{x}} \frac{\sigma x^H (B_u - B_v)}{1 - \mu(u - v)} > x \right). \end{aligned}$$

Since  $G_{u,v} := \sigma(B_u - B_v)/(1 - \mu(u - v))$  is a zero-mean continuous Gaussian process, Borel inequality implies that for  $x > \eta_1$

$$P(M_t^{H,-} > x) = P \left( \sup_{0 \leq u \leq v \leq t} (Y_u - Y_v) > x \right) \leq 2 \exp \left[ -\frac{1}{2} \frac{(x^{1-H} - \eta_x)^2}{\gamma} \right] \quad (8)$$

where  $\eta_x = E \left[ \sup_{0 \leq u \leq v \leq t/x} G_{u,v} \right]$ ,  $\gamma = \sup_{0 \leq u \leq v \leq t/x} E G_{u,v}^2$ , and it is observed that  $\eta_x \leq \eta_1$  holds for large  $x$ , in particular  $x > t$ . We have

$$\sup_{0 \leq u \leq v \leq t/x} G_{u,v} \stackrel{d}{=} \sup_{0 \leq u \leq v \leq t/x} \frac{\sigma B_{v-u}}{1 + \mu(v-u)} = \sup_{0 \leq v \leq t/x} \frac{\sigma B_v}{1 + \mu v}$$

and therefore

$$0 \leq \eta_x \leq \sup_{0 \leq v \leq t/x} \frac{\sigma}{1 + \mu v} E \left( \sup_{0 \leq v \leq t/x} B_v \right). \quad (9)$$

Note that  $\sigma/(1 + \mu v)$  is bounded from above by  $\sigma$  for large  $x$ . Since

$$\frac{\sqrt{2}(t/x)^H}{2\sqrt{\pi}} \leq E \left( \sup_{0 \leq v \leq t/x} B_v \right) \leq \frac{\sqrt{2}(t/x)^H}{\sqrt{\pi}},$$

we take  $\eta_x \propto (t/x)^H$  below in view of (9). On the other hand, when  $x$  is large enough, that is, when  $t/x < H/(1 - \mu H)$ ,

$$\gamma = \sup_{0 \leq u \leq v \leq t/x} E G_{u,v}^2 = \sup_{0 \leq v \leq t/x} \frac{\sigma^2 v^{2H}}{(1 + \mu v)^2} = \frac{\sigma^2 (t/x)^{2H}}{(1 + \mu t/x)^2}$$

is found. As a result of the bound (8), we get

$$\limsup_{x \rightarrow \infty} \frac{1}{x^2} \log P(M_t^{H,-} > x) \leq \lim_{x \rightarrow \infty} \left[ -\frac{1}{x^2} \frac{(x^{1-H} - (t/x)^H)^2}{2\sigma^2 (t/x)^{2H}} \right] = -\frac{1}{2\sigma^2 t^{2H}}$$

in view of  $H \in (0, 1)$ . □

## 5 Stronger Form of the Asymptotic Distribution

In this section, we directly find the asymptotical form of the tail probability using a characterization of [9] for Gaussian processes.

**Theorem 4** *Asymptotically, we have*

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(M_t^{H,-} > x)}{\Phi(x/t^{2H})} = 1.$$

when  $M_t$  is defined for standard fractional Brownian motion.

**Proof:** The proof is based on two conditions given in [9] that characterize the existence of the limiting distribution. We work with the mean zero Gaussian process  $\{B_u - B_v : 0 \leq u \leq v \leq t\}$ . Let  $\sigma_T^2 = \sup_{(u,v) \in T} \mathbb{E}(B_u - B_v)^2$ , which yields  $\sigma_T^2 = t^{2H}$ . The first condition is that there exists a unique  $(u_0, v_0)$  in  $T$  such that  $\mathbb{E}(B_{u_0} - B_{v_0})^2 = \sigma_T^2$ . This holds with  $(u_0, v_0) = (0, t)$ . For the second condition, the set

$$T_h = \{(u, v) \in T : \mathbb{E}(B_t(B_u - B_v)) \geq \sigma_T^2 - h^2\}$$



is defined for  $h > 0$ . In order to identify  $T_h$ , we find  $\mathbb{E}(B_t(B_u - B_v)) = \frac{1}{2}(v^{2H} - u^{2H} + (t - u)^{2H} - (t - v)^{2H})$ . Therefore,  $(u, v) \in T_h$  satisfy

$$v^{2H} - u^{2H} + (t - u)^{2H} - (t - v)^{2H} \geq 2t^{2H} - 2h^2 \quad (10)$$

This implies that (10) is satisfied by  $(\bar{u}, v) \in T_h$ , for fixed  $\bar{u}$ . Now, for  $\bar{u} > t/2$ , we have  $(t - \bar{u})^{2H} - u^{2H} < 0$ , and  $(t - v)^{2H} > 0$  for all  $v \in [0, t]$ . Then, from (10)

$$v^{2H} \geq 2t^{2H} - 2h^2 \geq t^{2H} - 2h^2.$$

On the other hand, for  $\bar{u} \leq t/2$ , we have  $(t - \bar{u})^{2H} \leq t^{2H}$ . Therefore, we get

$$v^{2H} + t^{2H} \geq v^{2H} - u^{2H} + (t - u)^{2H} - (t - v)^{2H} \geq 2t^{2H} - 2h^2$$

which again implies  $v^{2H} \geq t^{2H} - 2h^2$  for fixed  $\bar{u}$  and  $(\bar{u}, v) \in T_h$ . Since,  $f(v) = v^{2H}$  is convex, we get

$$v \geq t - Kh^2$$

for some constant  $K$  [9, pg.309]. Now, we consider second condition in [9] which requires

$$\lim_{h \rightarrow 0} h^{-1} \mathbb{E} \sup_{(u,v) \in T_h} B_u - B_v + B_t = 1 \quad (11)$$

In particular, we have

$$\mathbb{E} \sup_{(\bar{u}, v) \in T_h} B_{\bar{u}} - B_v + B_t \leq \mathbb{E} \sup_{v \geq \bar{u}, t-v \leq Kh^2} B_{\bar{u}} - B_v + B_t$$

Then, it follows that

$$\lim_{h \rightarrow 0} h^{-1} \mathbb{E} \sup_{v \geq \bar{u}, t-v \leq Kh^2} B_{\bar{u}} - B_v + B_t = \lim_{h \rightarrow 0} h^{-1} \mathbb{E} \sup_{t-v \leq Kh^2} B_{t-v} \quad (12)$$

since fractional Brownian motion has stationary increments and  $\mathbb{E}B_{\bar{u}} = 0$ . On the right hand side of (12), the supremum of fractional Brownian motion  $[0, Kh^2]$  is bounded by  $\sqrt{2}K^H h^{2H} / \sqrt{\pi}$  [3, 8] and hence we get the limit in (12) to be 0. Since  $T$  is separable, a monotone convergence argument extends the result for fixed  $\bar{u}$  to all  $T_h$  proving (11) [1, pg.47].  $\square$

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